Néron-Tate heights on algebraic curves and subgroups of the modular group

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Abstract. We give an expression for the value of the Néron-Tate height pairing of two divisors on an algebraic curve which involves special values of certain Dirichlet series associated to finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

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0.1. Introduction. Let X_E be any algebraic curve X defined over some number field E, then there is the Néron-Tate height pairing

$$\langle \cdot, \cdot \rangle_{NT} : Pic^0(X) \otimes Pic^0(X) \longrightarrow \mathbb{R}.$$

In this note we study the Néron-Tate height pairing of an arbitrary pair of degree zero divisors $D_1, D_2 \in Pic^0(X)$. Combining classical Arakelov theory and the extension given in [Kü] we are able to prove a new formula for $\langle D_1, D_2 \rangle_{NT}$. The main result of this note implies the following statement:

For any two divisors $D_1, D_2 \in Pic^0(X)$ their Néron-Tate height pairing $\langle D_1, D_2 \rangle_{NT}$ is the sum of the logarithm of an algebraic number and (up to 2π) a integral linear combination of special values of certain Dirichlet series, which come from non-holomorphic Eisenstein series associated to finite index subgroups of $SL_2(\mathbb{Z})$.

0.2. The main object to study the image of the Néron-Tate height pairing is that these real numbers are conjecturally related to special values of "motivated" L-functions. The most important result of this general picture is

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the theorem of Gross and Zagier which relates the height of Heegner divisors to the derivative of the Hasse-Weil L-function of certain elliptic curves [GZ]. For more details on these conjectures due to Birch Swinnerton-Dyer, Tate, Deligne and Beilinson we refer to the book [RSS]. We note that using an extension of Arakelov theory allows to calculate similar height pairings on varieties of higher dimension.

0.3. In order to derive the above statement from our result, we use Belyi's theorem, that given a smooth, projective algebraic curve X defined over $\overline{\mathbb{Q}}$ and a finite set S of points on X, then there exist a Zariski open subset $U \subset X$ disjoint from S, a subgroup $\Gamma \in \Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ and an isomorphism

$$U \cong \Gamma \setminus \mathfrak{h}$$

which is defined over $\overline{\mathbb{Q}}$. We may choose S so, that it contains the support of our divisors D_1 and D_2. Therefore we have reduced our problem of understanding the image of the Néron-Tate height pairing of divisors with support in S on X to the study of cuspidal divisors, i.e. divisors of degree zero with support in the cusps, on (general) modular curves $X(\Gamma) = \overline{\Gamma \setminus \mathfrak{h}}$. One should remark that $\langle D_1, D_2 \rangle_{NT}$ does of course not depend on the choice of S giving us the Belyi uniformization, but the individual terms in our formula may do.

0.4. Let Γ be any finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let us denote by $C(\Gamma) \subset Pic^0(X(\Gamma))$ the subgroup of cuspidal divisors. Combining the Faltings-Hriljac theorem with the extension of Arakelov theory proposed in [Kü], we are able to give a formula for the Néron-Tate height pairing of any two cuspidal divisors $D_1, D_2 \in C(\Gamma)$, see Theorem 6.11 below.

0.5. Recall, if Γ is a congruence subgroup, then by the Manin-Drinfeld Theorem the order of $C(\Gamma)$ is finite. In other words $C(\Gamma)$ is in the kernel of the Néron-Tate height pairing. In an example we will show, that in the case of modular curves associated to congruence subgroups the individual terms involved in our formula need not to vanish. Therefore there must be an cancellation of these terms, which in turn gives information about the bad reduction of modular curves, see also example 7.1.

0.6. As A. J. Scholl has observed the finiteness of $C(\Gamma)$ is equivalent to the rationality of the periods of differentials of the third kind constructed by certain linear combinations of non-holomorphic Eisenstein series. Using this relation he proved the finiteness to be equivalent to the algebraicity of the Fourier coefficients of these Eisenstein series [Sc]. A fact which was used by V.K. Murty and D. Ramakrishnan to determine algebraicity of generalized

Ramanujan sums associated to a Belyi uniformization of the Fermat curves [MR]. Corollary 6.13 gives an alternative necessary criterion for finiteness of $C(\Gamma)$.

0.7. Leitfaden. In section one we recall the extension of Arakelov theory used in the sequel. In section two we recall some facts on semi-stable models of modular curves. Section three provides the basis for section four, which is the technical heart of this paper; there most of the analytical work is done. In the fifth section we combine our considerations and the main result is proved in section six. In the last section we give an illustrating example and suggestions for further research.

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1 Review of Arakelov theory for arithmetic surfaces

In order to fix notation we recall some aspects of Arakelov Theory. For more details we refer to [Kü] and the references therein.

1.1. Notation. In this note an arithmetic surface $\mathcal{X}$, is a regular scheme of dimension 2 together with a projective flat morphism $f : \mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_E$, where $\mathcal{O}_E$ is the ring of integers of a number field E. Further more we assume that the generic fiber $\mathcal{X}_E$ of f is geometrically irreducible. We denote by $\mathcal{X}_\infty(\mathbb{C})$ the smooth projective manifold $\coprod_{\sigma:E\to\mathbb{C}} \mathcal{X}_\sigma(\mathbb{C})$; note the complex conjugation F_∞ acts on $\mathcal{X}_\infty$. An arithmetic surface is said to be semi-stable if all the fibers of the morphism f are semi-stable. In other words, $\mathcal{X}$ is the minimal regular model of $\mathcal{X}_E$.

1.2. Definitions. Let $\mathcal{X}$ be an arithmetic surface and $\mathcal{X}_\infty$ as above; for a finite set $\mathcal{S}$ of points $S_1, ..., S_r$ of $\mathcal{X}_\infty$ denote by $\mathcal{Y}_\infty$ the open complex manifold $\mathcal{X}_\infty \setminus \mathcal{S}$. For $S_j \in \mathcal{S}$ and $\varepsilon > 0$, denote by $B_\varepsilon(S_j) \subseteq \mathcal{X}_\infty$ the open disk of radius ε centered at S_j; let t be a local parameter at S_j $(j = 1, ..., r)$. For a line bundle $\mathcal{L}$ on $\mathcal{X}$, a singular metric h on the induced complex line bundle $\mathcal{L}_\infty$ on $\mathcal{X}_\infty$ is called *hermitian, logarithmically singular (with respect to $\mathcal{S}$)*, if the following two conditions hold:

(a) h is a smooth, hermitian metric on $\mathcal{L}_\infty$ restricted to $\mathcal{Y}_\infty$, invariant under F_∞;

(b) for each $S_j \in \mathcal{S}$ and any section l of $\mathcal{L}$, there exist a real number α and a positive, continuous function φ defined on $B_\varepsilon(S_j)$ and smooth away from the origin such that the equality

$$\|l(t)\| = -\log(|t|^2)^\alpha \cdot |t|^{\mathrm{ord}_{S_j}(l)} \cdot \varphi(t)$$

holds for all $t \in B_\varepsilon(S_j) \setminus \{0\}$; furthermore, there exist positive constants β and ρ such that the inequalities

$$\left|\frac{\partial\varphi(t)}{\partial t}\right| \leq \frac{\beta}{|t|^{1-\rho}}, \quad \left|\frac{\partial\varphi(t)}{\partial \bar{t}}\right| \leq \frac{\beta}{|t|^{1-\rho}}, \quad \left|\frac{\partial^2\varphi(t)}{\partial t\,\partial \bar{t}}\right| \leq \frac{\beta}{|t|^{2-\rho}}$$

hold for all $t \in B_\varepsilon(S_j) \setminus \{0\}$.

We call a line bundle $\mathcal{L}$ on $\mathcal{X}$ equipped with a logarithmically singular metric h a *hermitian, logarithmically singular line bundle* and denote it by $\overline{\mathcal{L}} = (\mathcal{L}, h)$. To indicate the dependence of the quantities α (resp. φ, β, ρ) on $l, \overline{\mathcal{L}}$ and $S_j \in \mathcal{S}$, we write instead $\alpha_{\overline{\mathcal{L}},j}$ (resp. $\varphi_{\overline{\mathcal{L}},j}, \beta_{\overline{\mathcal{L}},j}, \rho_{\overline{\mathcal{L}},j}$).

Two hermitian, logarithmically singular line bundles $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ on $\mathcal{X}$ are *isomorphic*, if

$$\overline{\mathcal{L}} \otimes \overline{\mathcal{M}}^{-1} \cong (\mathcal{O}_{\mathcal{X}}, |\cdot|).$$

The *generalized arithmetic Picard group*, denoted by $\widehat{\mathrm{Pic}}(\mathcal{X}, \mathcal{S})$, is the group of isomorphy classes of hermitian, logarithmically singular line bundles $\overline{\mathcal{L}}$ on $\mathcal{X}$ the group structure being given by the tensor product. Note, if $\mathcal{S} = \emptyset$, then $\widehat{Pic}(\mathcal{X}, \emptyset)$ coincides with the clasical arithmetic Picard group $\widehat{Pic}(\mathcal{X})$.

1.3. Definition. Let $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ be two hermitian, logarithmically singular line bundles on $\mathcal{X}$ and l, m (resp.) be non-trivial, global sections, whose induced divisors on $\mathcal{X}_\infty$ have no points in common. Then, the *generalized arithmetic intersection number* $\overline{\mathcal{L}}.\overline{\mathcal{M}}$ *of* $\overline{\mathcal{L}}$ *and* $\overline{\mathcal{M}}$ is given by

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} := (l.m)_{\mathrm{fin}} + \langle l.m \rangle_\infty; \tag{1.4}$$

here $(l.m)_{\mathrm{fin}}$ is defined by Serre's Tor-formula, which for l, m having proper intersection specializes to

$$(l.m)_{\mathrm{fin}} = \sum_{x \in \mathcal{X}} \log \sharp \left(\mathcal{O}_{\mathcal{X},x}/(l_x, m_x)\right),$$

where l_x, m_x are the local equations of l, m respectively at the point $x \in \mathcal{X}$ and

$$\langle l.m\rangle_\infty =$$
$$-(\log\|m\|)\big[\mathrm{div}(l) - \sum_{j=1}^{r} \mathrm{ord}_{S_j}(l)\cdot S_j\big] + \sum_{j=1}^{r} \mathrm{ord}_{S_j}(l)\Big(\alpha_{\overline{\mathcal{M}},j} - \log(\varphi_{\overline{\mathcal{M}},j}(0))\Big) -$$
$$\lim_{\varepsilon\to 0}\left(\sum_{j=1}^{r} \mathrm{ord}_{S_j}(l)\cdot \alpha_{\overline{\mathcal{M}},j}\cdot \log(-\log\varepsilon^2) + \int_{\mathcal{X}_\varepsilon} \log\|l\|\cdot c_1(\overline{\mathcal{M}})\right). \tag{1.5}$$

Note, in formula (1.5) the points $P_i \in \mathcal{S}$ with $\alpha_i = 0$ behave like the metric where smooth. In [Kü] we proved

1.6. Proposition. *The formula* (1.4) *induces a bilinear, symmetric pairing*

$$\widehat{\mathrm{Pic}}(\mathcal{X},\mathcal{S}) \times \widehat{\mathrm{Pic}}(\mathcal{X},\mathcal{S}) \;\longrightarrow\; \mathbb{R}$$

extending the pairing of Arakelov.

1.7. Definition. Let $\widehat{Pic}^0(\mathcal{X},\mathcal{S}) \subset \widehat{Pic}(\mathcal{X},\mathcal{S})$ denote the subgroup generated by the hermitian line bundles $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ satisfying $\deg(\mathcal{L}|\mathcal{C}_{\mathfrak{p},l}) = 0$ for all irreducible components $\mathcal{C}_{\mathfrak{p},l}$ of the fiber $f^{-1}(\mathfrak{p})$, and $\mathrm{c}_1(\overline{\mathcal{L}}) = 0$. Let $\widehat{Pic}^0(\mathcal{X})$ be the corresponding subgroup of $\widehat{Pic}(\mathcal{X})$ considered in classical Arakelov theory.

1.8. Proposition. *We have an equality*

$$\widehat{Pic}^0(\mathcal{X},\mathcal{S}) = \widehat{Pic}^0(\mathcal{X}). \tag{1.9}$$

Proof. Since $\mathrm{c}_1(\overline{\mathcal{L}}) = 0$, we note for an element $(\mathcal{L}, \|\cdot\|) \in \widehat{Pic}^0(\mathcal{X},\mathcal{S})$ that by [Kü], proposition 3.3, together with [Gr], formula (3.4), the hermitian metric $\|\cdot\|$ is in fact smooth on $\mathcal{X}_\infty$ and unique upto multiplication by a scalar. Therefore, $\widehat{Pic}^0(\mathcal{X},\mathcal{S})$ does not depend on $\mathcal{S}$ and hence coincides with $\widehat{Pic}^0(\mathcal{X})$. $\square$

To ease notation we will work in the sequel with $\mathbb{Q}$ coefficients, for this we put $\widehat{Pic}(\mathcal{X},\mathcal{S})_{\mathbb{Q}} = \widehat{Pic}(\mathcal{X},\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and analogousely $\widehat{Pic}^0(\mathcal{X})_{\mathbb{Q}}$.

1.10. Proposition. *Let D be a divisor on X_E with $\deg D = 0$. Then there exist a divisor $\mathcal{D}$, which may have rational coefficients, on $\mathcal{X}$ satisfying $\mathcal{D}_E = D$ and a hermitian metric $\|\cdot\|$ on $\mathcal{O}(D)_\infty$ such that*

$$\overline{\mathcal{O}(\mathcal{D})} = (\mathcal{O}(\mathcal{D}), \|\cdot\|) \in \widehat{Pic}^0(\mathcal{X})_{\mathbb{Q}}.$$

Moreover $\mathcal{D}$ is unique upto multiples of the fibers of f and the metric $\|\cdot\|$ is unique upto multiplication by a skalar.

Proof. For the existence of $\mathcal{D}$ we refer to lemme 6.14.1 in [MB], the existence of $\|\cdot\|$ follows from formula (3.4) in [Gr]. □

1.11. Proposition. (Faltings-Hriljac) *Let $\mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_E$ be a semi-stable arithmetic surface. Let D_1, D_2 be divisors on X_E with $\deg D_1 = \deg D_2 = 0$. Then, for all extensions $\overline{\mathcal{O}(\mathcal{D}_1)}, \overline{\mathcal{O}(\mathcal{D}_2)}$ of $\mathcal{O}(D_1)$, $\mathcal{O}(D_2)$ to $\widehat{Pic}^0(\mathcal{X})_{\mathbb{Q}}$, there is an equality*

$$-\langle D_1, D_2 \rangle_{NT} = \frac{1}{[E:\mathbb{Q}]} \cdot \overline{\mathcal{O}(\mathcal{D}_1)}.\overline{\mathcal{O}(\mathcal{D}_2)}, \tag{1.12}$$

where $\langle D_1, D_2 \rangle_{NT}$ is the Néron-Tate height pairing of the induced classes in the Picard group of X_E.

Proof. By our definition of $\widehat{Pic}^0(\mathcal{X})_{\mathbb{Q}}$ the statement follows immediately from the Faltings-Hriljac formula (see [Fa], [MB]). □

1.13. Remark. In our description of Arakelov theory we considered metrized line bundles, but as it is well known we could also take the point of view of considering divisors and Green's functions. Then hermitian, logarithmically singular line bundles correspond to divisors together with a Green's function having log-log singularities.

2 On semi-stable models for modular curves

2.1. Notations. Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index and $\mathfrak{h}$ be the upper half plane on which Γ acts by fractional linear transformation. The open Riemann surface $Y(\Gamma) = \Gamma \setminus \mathfrak{h}$ can be compactified by adding the finite set $\Gamma \setminus \mathbb{P}^1(\mathbb{Q})$ of cusps. The resulting compact Riemann surface will be denoted by $X(\Gamma)$ and called a *(general) modular curve.* We are interested in the *group of cuspidal divisors* $C(\Gamma)$, which is generated by divisors on $X(\Gamma)$ of degree zero with support in the cusps.

2.2. As it is well known the modular curve $X(\Gamma)$ has a model $X(\Gamma)_E$ defined over some number field E and therefore a semi-stable model

$$f : \mathcal{X}(\Gamma) \longrightarrow \operatorname{Spec} \mathcal{O}_E \tag{2.3}$$

exist. It is unique if the genus is bigger than one. To ease notations we will denote $\mathcal{X}(\Gamma)$ in the sequel by $\mathcal{X}$.

2.4. The curves $X(\Gamma)$ under consideration come together with a natural Belyi morphism $X(\Gamma)_E \longrightarrow X(1)_{\mathbb{Q}}$, we use this structure to obtain some properties of the semi-stable model over the ring of integers of E. Recall the natural model for $X(1)$ is $\mathbb{P}^1_{\mathbb{Z}}$ where the embedding is given by the modular forms $j \cdot \Delta$ and Δ. Define $X(\Gamma)_{\mathcal{O}_E}$ to be the normalization of $\mathcal{X}(1) \times_{\mathbb{Z}} \mathcal{O}_E$ in $X(\Gamma)_E$. A prime $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_E$ is called a prime of bad reduction if $f_\Gamma^{-1}(\mathfrak{p})$ is a fiber of bad reduction for the morphism $f_\Gamma : X(\Gamma)_{\mathcal{O}_E} \longrightarrow \operatorname{Spec} \mathcal{O}_E$, i.e., $f_\Gamma^{-1}(\mathfrak{p}) = \bigcup_l \mathcal{C}_{\mathfrak{p},l}$ is a non-smooth fiber.

2.5. Definition. We denote by $\mathcal{P}_\Gamma$ the set of primes $\mathfrak{p}$ of bad reduction of $X(\Gamma)_{\mathcal{O}_E}$. We denote by b_Γ the smallest natural number with $(b_\Gamma) \subseteq \mathfrak{p}$ for all primes $\mathfrak{p} \in \mathcal{P}_\Gamma$.

2.6. In order to get a semi-stable model $\mathcal{X}(\Gamma)$ for $X(\Gamma)_{\mathcal{O}_E}$, we have to iterate blow-ups of the singularities of $X(\Gamma)_{\mathcal{O}_E}$ and blow-downs of the contained (-1) curves [Li]. Note this process does not increase the set of primes of bad reduction. For later use we remark that the proper morphism

$$\mathcal{X}(\Gamma)_{\mathcal{O}_E[1/b_\Gamma]} \longrightarrow \mathcal{X}(1)_{\mathcal{O}_E[1/b_\Gamma]},$$

does in general not extend to a proper morphism of schemes defined over $\operatorname{Spec} \mathcal{O}_E$.

2.7. Definition. The cusps of $X(\Gamma)$ are by construction algebraic points and we denote by s_j the Zariski closure of the cusp $S_j \in X(\Gamma)(\overline{\mathbb{Q}})$ in the scheme $\mathcal{X}(\Gamma)$. We denote by $\mathcal{P}_\mathcal{S}$ the set of primes $\mathfrak{p}$ for which there exist two different cusps S_j and S_k such that

$$s_j \cap f^{-1}(\mathfrak{p}) = s_k \cap f^{-1}(\mathfrak{p}).$$

2.8. Remarks. a.) For Γ a subgroup of $\Gamma(2)$ one may describe the primes of bad reduction as divisors of the order of a certain subgroup of S_n, where

$n = [\Gamma(2) : \Gamma]$ (see [Bi]). Can one show that their product equals the level, i.e. the least common multiple of the widths b_j of the cusps S_j of $X(\Gamma)$; or in other words the least common multiple of the ramification orders above the cusp S_∞ of $X(1)$?

b.) Does for arbitrary $\Gamma \subset \Gamma(1)$ the inclusion $\mathcal{P}_\mathcal{S} \subset \mathcal{P}_\Gamma$ hold?

3 On non-holomorphic Eisenstein series

We first recall some facts on the theory of Eisenstein series the standard reference is [Ku].

3.1. Complex structure. Let $\mathcal{S} = \{S_1 = \infty, \ldots, S_h\}$ be a complete set of cusps for $X(\Gamma)$. For each $S_j \in \mathcal{S}$ we let Γ_j be its stabilizer in Γ. We fix also an element $\sigma_j \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma_j(\infty) = S_j$ and

$$\sigma_j^{-1}\Gamma_j\sigma_j = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \middle| m \in \mathbb{Z} \right\};$$

note there exist an element $\gamma_j \in \Gamma(1)$ such that

$$\sigma_j = \begin{pmatrix} \sqrt{b_j} & 0 \\ 0 & 1/\sqrt{b_j} \end{pmatrix} \cdot \gamma_j,$$

where b_j is the width of the cusp S_j. Let $\tau = x+iy \in \mathfrak{h}$, then a local parameter for the cusp S_j, considered as a point on the compact Riemann surface $X(\Gamma)$, is given by $t_j = \exp(2\pi i\sigma_j^{-1}(\tau))$; for more details on the complex structure of $X(\Gamma)$ we refer to [Kü].

3.2. Definition. For each cusp S_j there is a *non-holomorphic Eisenstein series* $E_j(\tau; s)$, which, for $s \in \mathbb{C}$, $\mathrm{Re}s > 1$, is defined by the convergent series

$$E_j(\tau; s) = \sum_{\sigma\in\Gamma_j\backslash\Gamma} \mathrm{Im}\left(\sigma_j^{-1}\sigma(\tau)\right)^s.$$

3.3. Properties. Let us recall some properties of these functions which will need later on: For all $S_j \in \mathcal{S}$ the function $E_j(\tau; s)$ has a meromorphic continuation to the s-plane, with a simple pole in $s = 1$ with residue $3/(\pi \cdot [\Gamma(1) : \Gamma])$. For all $\gamma \in \Gamma$ we have $E_j(\gamma(\tau); s) = E_j(\tau; s)$. We have

$$\triangle E_j(\tau; s) = s(s-1)E_j(\tau; s),$$

where $\triangle = y^2\,(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ is the hyperbolic Laplacian. The Fourier expansion of $E_j(\tau; s)$ at the cusp S_k is given by

$$E_j(\sigma_k(\tau; s) = \sum_{n\in\mathbb{Z}} a_{jk,n}(y; s)\exp(2\pi i x),$$

where

$$\begin{aligned} a_{jk,0}(y; s) &= \delta_{j,k}\cdot y^s + \phi_{jk,0}(s)\pi^s \frac{\Gamma(s-1/2)}{\Gamma(s)}\cdot y^{1-s} \\ a_{jk,n}(y; s) &= \phi_{jk,n}(s) K_s(2\pi|n|y) \qquad (n \neq 0); \end{aligned}$$

here $\Gamma(s)$ is the gamma function, $K_s(t)$ the K-Bessel function of rapid decay and

$$\phi_{jk,n} = \sum_{\sigma\in\Gamma_j\backslash\Gamma/\Gamma_k} \frac{\exp(2\pi i n d/c)}{|c|^{2s}},$$

with $c>0$, $d \mod c$, $\sigma_j\sigma\sigma_k = \begin{pmatrix} * & * \\ c & d\end{pmatrix}$. Finally the *scattering matrix*

$$\Phi_\Gamma(s) = \left(\pi^s\frac{\Gamma(s-1/2)}{\Gamma(s)}\cdot\phi_{jk,0}(s)\right)_{j,k}$$

is symmetric and satisfies the functional equation $\Phi_\Gamma(s)\cdot\Phi_\Gamma(1-s) = (\delta_{j,k})$. Note all the coefficients of the scattering matrix are Dirichlet series in a general sense, having a meromorphic continuation with a simple pole in $s=1$ of residue $3/(\pi\cdot[\Gamma(1):\Gamma])$.

3.4. Definition. For all pairs j,k we define the numbers C_{jk} to be the constant term at 1 of the Dirichlet series $(\Phi_\Gamma)_{jk}(s)$, i.e.,

$$C_{jk} := \mathrm{CT}_{s=1}\left(\pi^s\frac{\Gamma(s-1/2)}{\Gamma(s)}\cdot\phi_{jk,0}(s)\right). \tag{3.5}$$

3.6. Remark. If Γ is a congruence subgroup of certain type, then the functions $(\Phi_\Gamma)_{jk}(s)$ are determined by explicit formulas see e.g. [He] or example 7.1. For arbitrary Γ, at least to the knowledge of the author, there are no explicit formulas for these functions in the literature.

4 More on non-holomorphic Eisenstein series

Non-holomorphic Eisenstein series occur in various branches of mathematics. Our point of view is that these functions give rise to Green's functions satisfying log-log growth conditions.

4.1. Recall in the particular case $\Gamma = \Gamma(1)$ there is the well-known Kronecker limit formula

$$4\pi \lim_{s\to 1} \big(E(\tau;s) - \Phi_{\Gamma(1)}(s)\big) - 12\log(4\pi) = -\log\|\Delta\|^2_{Pet},$$

where Δ is the unique cusp form of weight 12 and $\|\cdot\|^2_{Pet}$ is the Petersson metric on the line bundle of modular forms. The Petersson metric is a hermitian, logarithmically singular metric with curvature form given by the $\mathrm{SL}_2(\mathbb{R})$-invariant $(1,1)$-form

$$\omega_{\Gamma(1)} := \frac{dxdy}{4\pi y^2}$$

associated to the hyperbolic metric on $\mathfrak{h}$. Note $\omega_{\Gamma(1)}$ induces a logarithmically singular volume form on $X(1) := X(\Gamma(1))$ of volume $1/12$. Considering Δ as a function on $X(1)$ it has divisor $\operatorname{div}\Delta = S_\infty$ and the local expansion at the cusp S_∞ is given by

$$-\log\|\Delta\|^2_{Pet} = -\log|t|^2 - 12\log(-\log|t|^2) + f_\infty(t)$$

with some continuous function $f_\infty(t)$, smooth outside the elliptic fixed points, such that $f_\infty(0) = 0$. Furthermore, on $X(1)$ the equality

$$dd^c - \log\|\Delta\|^2_{Pet} + \delta_{S_\infty} = 12\cdot\omega_{\Gamma(1)}$$

holds. We say $-\log\|\Delta\|^2_{Pet}$ is a *hyperbolic Green's function for the cusp* S_∞.

4.2. Definition. Let Γ be a finite index subgroup of $\Gamma(1)$. Let S_j be a cusp of $X(\Gamma)$, then we define the function $g_j(\tau)$ by

$$g_j(\tau) := 4\pi \lim_{s\to 1} \Big(E_j(\tau;s) - (\Phi_\Gamma)_{jj}(s)\Big) - \frac{12}{[\Gamma(1):\Gamma]}\log(4\pi), \tag{4.3}$$

and we denote by g_j the function on $X(\Gamma)$ induced by $g_j(\tau)$.

4.4. Proposition. *The function g_j is invariant under complex conjugation and satisfies the equality*

$$dd^c g_j + \delta_{S_j} = \frac{12}{[\Gamma(1):\Gamma]}\cdot\omega_{\Gamma(1)}. \tag{4.5}$$

We say g_j is a *hyperbolic Green's function for the cusp* S_j.

Proof. The Fourier expansion of $E_j(\tau; s)$ implies that for $\tau \in \mathfrak{h}$ the Fourier expansion of $g_j(\tau)$ at S_j is given by

$$g_j(\sigma_j(\tau)) = 4\pi y - \frac{12}{[\Gamma(1):\Gamma]}\log(4\pi y) + \sum_{m\neq 0} a_{jj,m}(y;1)\exp(2\pi i m x).$$

At the cusp S_k it is given by

$$g_j(\sigma_k(\tau)) = \frac{-12}{[\Gamma(1):\Gamma]}\log(4\pi y) + 4\pi(C_{jk} - C_{jj}) + \sum_{m\neq 0} a_{jj,m}(y;1)\exp(2\pi i m x).$$

Now, using the identity $K_{1/2}(x) = \sqrt{\pi/(2x)}\exp(-x)$, we get that $g_j(\tau)$ is a smooth function on $\mathfrak{h}$ which is invariant under the complex conjugation F_∞. Furthermore, since $E_j(\tau)$ is an eigenfunction of the hyperbolic Laplacian, we derive that for all $\tau \in \mathfrak{h}$ the equality

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) g_j(\tau) = \frac{12}{[\Gamma(1):\Gamma]}\cdot\frac{1}{4\pi y^2}$$

holds. At the cusps S_j and S_k we have in the corresponding local parameter

$$g_j(t_j) = -\log|t_j|^2 - \frac{12}{[\Gamma(1):\Gamma]}\log(-\log|t_j|^2) + f_{jj}(t_j), \tag{4.6}$$

$$g_j(t_k) = -\frac{12}{[\Gamma(1):\Gamma]}\log(-\log|t_k|^2) + f_{jk}(t_k), \tag{4.7}$$

where f_{jj} and f_{jk} are continuous functions, smooth outside outside the elliptic fixed points. They have the special values

$$f_{jj}(0) = 0$$
$$f_{jk}(0) = 4\pi\left(C_{jk} - C_{jj}\right).$$

The claim follows from the above description of g_j in local coordinates. □

For later use we state

4.8. Lemma. *Let $B_\varepsilon(S_j) = \{P \in X(\Gamma) \mid |t_j(P)| < \varepsilon\}$ be a small ε-neighbourhood of S_j. Then there is an equality*

$$\lim_{\varepsilon\to 1}\left(\log(-\log|\varepsilon|^2) - \int_{X(\Gamma)\setminus B_\varepsilon(S_j)} g_j\,\omega_{\Gamma(1)}\right) = \frac{[\Gamma(1):\Gamma]}{12}\cdot 4\pi C_{jj} + 2\log(4\pi). \tag{4.9}$$

Proof. Let us fix $\varepsilon = \exp(-2\pi T)$, $T >> 0$ and set $X_\varepsilon = X(\Gamma) \setminus B_\varepsilon(S_j)$. Let us choose a fundamental domain $\mathcal{F}_\Gamma$ for the action of Γ on $\mathfrak{h}$, then after possible conjugation with σ_j a pre-image of $B_\varepsilon(S_j)$ is given by the set

$$\mathcal{F}_{\sigma_j,\varepsilon} = \{x + iy \in \mathfrak{h} \,|\, y > T,\, 0 \leq x < 1\}.$$

As $E_j(s)$ is an eigenfunction of the hyperbolic Laplacian we get by means of Green's formula

$$\begin{aligned}
\int_{X_\varepsilon} 4\pi E_j(s)\,\omega_{\Gamma(1)} &= \frac{1}{s(s-1)} \int_{X_\varepsilon} \triangle E_j(s)\, \frac{dxdy}{y^2} \\
&= \frac{1}{s(s-1)} \int_{\partial B_\varepsilon(S_j)} \frac{\partial E_j(s)}{\partial \eta} dl \\
&= \frac{1}{s(s-1)} \int_0^1 \frac{\partial E_j(\sigma_j^{-1}(\tau); s)}{\partial y} dx \bigg|_{y=T} \\
&= \frac{T^{s-1}}{s-1} - (\Phi_\Gamma)_{jj}(s) \cdot \frac{T^{-s}}{s}
\end{aligned}$$

Then, using the Laurent expansion of $(\Phi_\Gamma)_{jj}(s)$ we get

$$\begin{aligned}
\int_{X_\varepsilon} 4\pi E_j(s)\,\omega_{\Gamma(1)} = &\left(1 - \frac{3}{\pi[\Gamma(1):\Gamma]} \cdot \frac{1}{T}\right)(s-1)^{-1} \\
&+ \log(T) + O\left(\frac{\log T}{T}\right) + O(s-1).
\end{aligned}$$

Furthermore, since we have

$$\int_{X_\varepsilon} \omega_{\Gamma(1)} = \left(\frac{[\Gamma(1):\Gamma]}{12} - \frac{1}{4\pi T}\right),$$

we easily determine

$$\begin{aligned}
&\int_{X_\varepsilon} \left(4\pi \left(E_j - (\Phi_\Gamma)_{jj}(s)\right) - \frac{12\log(4\pi)}{[\Gamma(1):\Gamma]}\right) \omega_{\Gamma(1)} \\
&\quad = \left(\left(1 - \frac{3}{\pi[\Gamma(1):\Gamma]} \cdot \frac{1}{T}\right) - \left(\frac{[\Gamma(1):\Gamma]}{12} - \frac{1}{4\pi T}\right) \cdot \frac{12}{[\Gamma(1):\Gamma]}\right)(s-1)^{-1} \\
&\qquad - \frac{[\Gamma(1):\Gamma]}{12} \cdot 4\pi \cdot C_{jj} + \log(T) - \log(4\pi) + O\left(\frac{\log T}{T}\right) + O(s-1).
\end{aligned}$$

Resorting the above terms and taking the limit $s \to 1$ leads to

$$\log(4\pi T) - \int_{X_\varepsilon} g_j \, \omega_{\Gamma(1)} = \frac{[\Gamma(1):\Gamma]}{12} \cdot 4\pi \cdot C_{jj} + 2\log(4\pi) + O\left(\frac{\log T}{T}\right).$$

Now the claimed formula follows as $4\pi T = -\log|\varepsilon|^2$. □

4.10. Remark. Similar calculations maybe found in [Ku], page 19; compare also the special case $\Gamma = \Gamma(1)$ in [Za] and its application in [Kü].

5 Key-identity

5.1. By construction we have $\mathcal{X}_\infty = \coprod_{\sigma: E \to \mathbb{C}} \mathcal{X}_\sigma(\mathbb{C})$, where each component $\mathcal{X}_\sigma(\mathbb{C})$ is isomorphic to $X(\Gamma)$. Let us denote by $\widehat{Pic}(\mathcal{X}, \mathcal{S})$ the group of hermitian, logarithmically singular line bundles, having singularities as in definition 1.2 at most at the cusps on each copy of $X(\Gamma)$ and at the elliptic points on each copy of $X(\Gamma)$. For each cusp S_j we metrize the line bundle $\mathcal{O}(S_j)$ on $X(\Gamma)$ by defining the norm of the canonical section by

$$\|1_{S_j}\|_{hyp} = \exp - g_j.$$

We obtain a metric on $\mathcal{O}(s_j)_\infty$ over $\mathcal{X}_\infty$ by metrizing each component with the above metric. We put

$$\overline{\mathcal{O}(s_j)} = (\mathcal{O}(s_j), \|\cdot\|_{hyp}).$$

5.2. Proposition. a) *We have* $\overline{\mathcal{O}(s_j)} \in \widehat{Pic}(\mathcal{X}, \mathcal{S})$.

b) *Let* $D = \sum_j n_j S_j$ *be a cuspidal divisor. Then, there exist rational numbers* $m_{\mathfrak{p},l}$, *non-zero at most for* $\mathfrak{p} \in \mathcal{P}_\Gamma$ *such that the class of*

$$\overline{\mathcal{O}(\mathcal{D})} = \bigotimes_{\mathfrak{p} \in \mathcal{P}_\Gamma} \mathcal{O}\left(\mathcal{C}_{\mathfrak{p},l}\right)^{\otimes m_{\mathfrak{p},l}} \bigotimes_j \overline{\mathcal{O}(s_j)}^{\otimes n_j} \tag{5.3}$$

in $\widehat{Pic}(\mathcal{X}, \mathcal{S})_\mathbb{Q}$ *is an extension of* $\mathcal{O}(D)$ *to* $\widehat{Pic}^0(\mathcal{X})_\mathbb{Q}$.

Proof. a) Using the local descriptions (4.6), (4.7) of g_j given in proposition 4.4 we derive that $\|\cdot\|_{hyp}$ is a hermitian, logarithmically singular metric. More precisely this metric is smooth outside the set of cusps and the set of elliptic points on each copy of $X(\Gamma)$. For all cusps we have $\alpha = \frac{12}{[\Gamma(1):\Gamma]}$ and for all elliptic points we have $\alpha = 0$.

b) By proposition 1.10 the divisor D extends to a divisor $\mathcal{D}$ with rational coefficients on $\mathcal{X}$, which has zero degree on all irreducible components of the fibers of f. In particular there exist rational numbers $m_{\mathfrak{p},l}$, non-zero at most for $\mathfrak{p} \in \mathcal{P}_\Gamma$ such that

$$\mathcal{O}(\mathcal{D}) = \bigotimes_{\mathfrak{p}\in\mathcal{P}_\Gamma} \mathcal{O}\left(\mathcal{C}_{\mathfrak{p},l}\right)^{\otimes m_{\mathfrak{p},l}} \bigotimes_j \mathcal{O}(s_j)^{n_j}.$$

It remains to show that for the right hand side of (5.3) the first Chern form vanishes, but since $\sum_j n_j = 0$ this follows from proposition 4.4. $\square$

5.4. Remark. The fact that $\sum_j m_j g_j$ is a Green's function for the cuspidal divisor $D = \sum_j m_j S_j$ is due to A. Scholl [Sc] and was the starting point of this paper. Using this fact he proved the following statements to be equivalent

1. Some non-zero multiple of D is a principal divisor on $X(\Gamma)$.
2. For all natural numbers $l > 0$ the numbers $4\pi^2 l \cdot \sum_j m_j \phi_{jk,l}(1)$, where the functions $\phi_{jk,l}(s)$ are given by the Fourier expansion at S_k of the Eisenstein series $E_j(\tau; s)$, are algebraic numbers.

5.5. Theorem. *Let $D_1 = \sum_j n_j S_j, D_2 = \sum_k m_k S_k$ be two cuspidal divisors and let $\overline{\mathcal{O}(\mathcal{D}_1)}$ and $\overline{\mathcal{O}(\mathcal{D}_2)}$ be their extension to $\widehat{Pic}^0(\mathcal{X})_{\mathbb{Q}}$ provided by proposition 5.2. Then there exist rational numbers $a_{\mathfrak{p}}$ such that*

$$-\langle D_1, D_2\rangle_{NT} = \sum_{\mathfrak{p}\in\mathcal{P}_\Gamma} a_p \log N(\mathfrak{p})$$
$$+ \frac{1}{[E:\mathbb{Q}]}\left(\sum_j n_j m_j \overline{\mathcal{O}(s_j)}.\overline{\mathcal{O}(s_j)} + \sum_{j\neq k} n_j m_k \overline{\mathcal{O}(s_j)}.\overline{\mathcal{O}(s_k)}\right).$$

Proof. Applying the Faltings-Hriljac formula 1.11 we derive

$$
\begin{aligned}
-\langle D_1, D_2\rangle_{NT} &= \frac{1}{[E:\mathbb{Q}]}\cdot \overline{\mathcal{O}(\mathcal{D}_1)}.\overline{\mathcal{O}(\mathcal{D}_2)} \\
&= \frac{1}{[E:\mathbb{Q}]}\left(\bigotimes_{\mathfrak{p}\in\mathcal{P}_\Gamma}\mathcal{O}\left(f^{-1}(\mathfrak{p})^{(l)}\right)^{\otimes n_{\mathfrak{p}}^{(l)}}\bigotimes_j \overline{\mathcal{O}(s_j)}^{\otimes n_j}\right). \\
&\qquad \left(\bigotimes_{\mathfrak{p}\in\mathcal{P}_\Gamma}\mathcal{O}\left(f^{-1}(\mathfrak{p})^{(l)}\right)^{\otimes m_{\mathfrak{p}}^{(l)}}\bigotimes_k \overline{\mathcal{O}(s_j)}^{\otimes m_k}\right) \\
&= \sum_{\mathfrak{p}\in\mathcal{P}_\Gamma} a_{\mathfrak{p}}\log \mathrm{N}(\mathfrak{p}) \\
&\quad + \frac{1}{[E:\mathbb{Q}]}\left(\sum_j n_j m_j \overline{\mathcal{O}(s_j)}.\overline{\mathcal{O}(s_j)} + \sum_{j\neq k} n_j m_k \overline{\mathcal{O}(s_j)}.\overline{\mathcal{O}(s_k)}\right).
\end{aligned}
$$

For the last equality we used the bilinearity of the generalized arithmetic intersection number. □

5.6. Remarks. a.) Although the left hand side in theorem 5.5 is defined in classical terms the right hand side is only well-defined applying the extension of Arakelov theory presented in section 1 .

b.) Note, given for all primes $\mathfrak{p}$ the intersection matrix of the irreducible components of the fiber $f^{-1}(\mathfrak{p})$ and the knowledge which irreducible component intersects the horizontal divisor s_j then one can calculate the multiplicities $a_{\mathfrak{p}}$.

6 Main Theorem

We will now calculate the generalized arithmetic intersection numbers occurring in the last equality of theorem 5.5, which in turn provides our main theorem.

6.1. Proposition. *There exist rational numbers $\alpha_{\mathfrak{p}}$ such that*

$$
\begin{aligned}
\frac{1}{[E:\mathbb{Q}]}\cdot\overline{\mathcal{O}(s_j)}.\overline{\mathcal{O}(s_k)} &= \sum_{\mathfrak{p}\in\mathcal{P}_S}\alpha_{\mathfrak{p}}\log N(\mathfrak{p}) + 2\pi\left(C_{jk}-C_{kk}-C_{jj}\right) \\
&\quad + \frac{12}{[\Gamma(1):\Gamma]} - \frac{24\log(4\pi)}{[\Gamma(1):\Gamma]}.
\end{aligned} \tag{6.2}
$$

Proof. Using definition (1.4) we have $\overline{\mathcal{O}(s_j)}.\overline{\mathcal{O}(s_j)} = (s_j, s_k)_{\text{fin}} + (s_j, s_k)_\infty$. As two different cusps meet at most in the fibers above $\mathfrak{p} \in \mathcal{P}_S$ we know that there exist some $\alpha_\mathfrak{p} \in \mathbb{Q}$ with

$$(s_j, s_k)_{\text{fin}} = [E : \mathbb{Q}] \cdot \sum_{\mathfrak{p} \in \mathcal{P}_S} \alpha_\mathfrak{p} \log \mathrm{N}(\mathfrak{p}). \tag{6.3}$$

Since all components of $\mathcal{O}(s_j)_\infty$ and $\mathcal{O}(s_k)_\infty$ are metrized the same way, we have to perform the calculations only in one of the $[E : \mathbb{Q}]$ components of $\mathcal{X}_\infty$. Note, all the data needed for formula (1.5) is presented in the proof of 5.2 and the remaining integration is done in lemma 4.8. Therefore, we derive for the second term

$$\begin{aligned} \frac{2}{[E:\mathbb{Q}]} \cdot (s_j, s_k)_\infty = 0 + 1 \cdot \Big(\frac{12}{[\Gamma(1):\Gamma]} + f_{jk}(0) \Big) - \\ \lim_{\varepsilon \to 0} \left(1 \cdot \frac{12}{[\Gamma(1):\Gamma]} \cdot \log(-\log \varepsilon^2) - \int_{X_\varepsilon} g_j \cdot \frac{12}{[\Gamma(1):\Gamma]} \cdot \omega_{\Gamma(1)} \right) \\ = 4\pi \left(C_{jk} - C_{kk} - C_{jj} \right) + \frac{12}{[\Gamma(1):\Gamma]} - \frac{24 \log(4\pi)}{[\Gamma(1):\Gamma]}. \end{aligned} \tag{6.4}$$

Adding the two quantities (6.3) and (6.4) gives the claim. □

6.5. Remark. Note, given for all primes $\mathfrak{p}$ the intersection matrix of the irreducible components of the fiber $f^{-1}(\mathfrak{p})$ and the intersection behaviour of the divisors s_j with s_k then one can calculate the multiplicities $\alpha_\mathfrak{p}$.

6.6. Proposition. *There exist rational numbers $\beta_\mathfrak{p}$ such that*

$$\frac{1}{[E:\mathbb{Q}]} \cdot \overline{\mathcal{O}(s_j)}.\overline{\mathcal{O}(s_j)} = \sum_{\mathfrak{p} \in \mathcal{P}_\Gamma \cup \mathcal{P}_S} \beta_\mathfrak{p} \log N(\mathfrak{p}) - 2\pi C_{jj} + \frac{12}{[\Gamma(1):\Gamma]} - \frac{24 \log(4\pi)}{[\Gamma(1):\Gamma]}. \tag{6.7}$$

Proof. In order to get proper intersection on the generic fiber of $\mathcal{X}$ we move s_j, the divisor of the section 1_{s_j} of $\mathcal{O}(s_j)$ by the divisor $\operatorname{div} h$ of a rational function h. For this we will use the morphism $\pi : \mathcal{X} \longrightarrow \mathcal{X}(1)_{\mathcal{O}_E[1/b_\Gamma]}$. Note by bilinearity of the generalized arithmetic intersection number we may work with $\mathbb{Q}$ coefficients. We choose the rational function h such that

$$h|_{\mathcal{X}_{\mathcal{O}_E[1/b_\Gamma]}} = \pi^*(j))^{\otimes 1/b_j},$$

where j is the j-function and b_j is the width of the cusp S_j. Let $\operatorname{div} h = \operatorname{div} h^+ + \operatorname{div} h^-$ be the decomposition of $\operatorname{div} h$ into its positive and negative part. Then by the projection formula we have on $\mathcal{X}_{\mathcal{O}_E[1/b_\Gamma]}$ that

$$(s_j, \operatorname{div} h^+)_{\mathrm{fin}, \mathcal{X}_{\mathcal{O}_E[1/b_\Gamma]}} = (\infty, 0)_{\mathrm{fin}, \mathbb{P}^1_{\mathcal{O}_E[1/b_\Gamma]}} = 0,$$

therefore only the primes $\mathfrak{p} \in \mathcal{P}_\Gamma$ may give a contribution to the intersection number at the finite places $(s_j, \operatorname{div} h^+)_{\mathrm{fin}}$. As furthermore s_j and $s_j + \operatorname{div} h^-$ intersect at most in the fibers above $\mathfrak{p} \in \mathcal{P}_\Gamma \cup \mathcal{P}_S$, there exist some $\beta_\mathfrak{p} \in \mathbb{Q}$ with

$$(s_j, s_j + \operatorname{div} h)_{\mathrm{fin}} = [E : \mathbb{Q}] \cdot \sum_{\mathfrak{p} \in \mathcal{P}_\Gamma \cup \mathcal{P}_S} \beta_\mathfrak{p} \log \mathrm{N}(\mathfrak{p}). \tag{6.8}$$

Since each component of $\mathcal{O}(s_j)_\infty$ is metrized the same way, we have to perform the calculations only in one of the $[E : \mathbb{Q}]$ components of $\mathcal{X}_\infty$. Recall the well-known Fourier expansion of the j-function. Then we get with the help of 3.1 and (4.6) for the local expansion of the rational section $1_{s_j} \cdot h$ of $\mathcal{O}(s_j)$ at the cusp S_j the formula

$$-\log \|1_{s_j} \cdot h\|_{hyp} = \frac{-12}{[\Gamma(1) : \Gamma]} \log(-\log |t_j|^2) + f_{jj}(t_j) + O(t_j).$$

Using this modified data we proceed as in 6.1 and derive for the second term

$$\begin{aligned} \frac{2}{[E : \mathbb{Q}]} \cdot (s_j, s_j + \operatorname{div} h)_\infty &= 0 + 1 \cdot \Big(\frac{12}{[\Gamma(1) : \Gamma]} - 0 \Big) - \\ &\lim_{\varepsilon \to 0} \Big(1 \cdot \frac{12}{[\Gamma(1) : \Gamma]} \cdot \log(-\log \varepsilon^2) - \int_{\mathcal{X}_\varepsilon} g_j \cdot \frac{12}{[\Gamma(1) : \Gamma]} \cdot \omega_\Gamma \Big) \\ &= -4\pi C_{jj} + \frac{12}{[\Gamma(1) : \Gamma]} - \frac{24 \log(4\pi)}{[\Gamma(1) : \Gamma]}. \end{aligned} \tag{6.9}$$

Adding the above quantities (6.8) and (6.9) gives the claim. □

6.10. Remark. Note, given for all primes $\mathfrak{p}$ the intersection matrix of the irreducible components of the fiber $f^{-1}(\mathfrak{p})$ and the intersection behavior of the divisors s_j with all s_k then one can calculate the multiplicities $\beta_\mathfrak{p}$.

6.11. Theorem. *Let $D_1 = \sum_j n_j S_j, D_2 = \sum_k m_k S_k \in C(\Gamma)$ be a cuspidal divisors on a general modular curve $X(\Gamma)$ associated to a subgroup*

$\Gamma \in SL_2(\mathbb{Z})$ of finite index. Then we have the following formula for their Néron-Tate height pairing

$$-\langle D_1, D_2\rangle_{NT} = \sum_{\mathfrak{p}\in\mathcal{P}_\Gamma\cup\mathcal{P}_S} \gamma_p \log N(\mathfrak{p}) \\ -2\pi\left(\sum_j n_j m_j \cdot C_{jj} + \sum_{j\neq k} n_j m_k \left(C_{jj}+C_{kk}-C_{jk}\right)\right), \tag{6.12}$$

where all the numbers $\gamma_p \in \mathbb{Q}$ and the real numbers C_{jk} are given by the constant term of the generalized Dirichlet series

$$C_{jk} := \mathrm{CT}_{s\to 1}\left(\pi^s \frac{\Gamma(s-1/2)}{\Gamma(s)} \sum_{\sigma\in\Gamma_j\backslash\Gamma/\Gamma_k} \frac{1}{|c|^{2s}}\right), \qquad \sigma_j\sigma\sigma_k = \begin{pmatrix} * & * \\ c & * \end{pmatrix}.$$

Moreover if one has precise control on the semi-stable model then the rational numbers γ_p can be calculated explicitely.

Proof. A formula for the Néron-Tate height pairing in question is given in theorem 5.5. The generalized arithmetic intersection numbers involved in this formula are determined in proposition 6.1 and proposition 6.6. Collecting all the terms and using the fact $\sum_j n_j = \sum_k m_k = 0$ we derive our main result. □

6.13. Corollary. *Let $D = \sum_k m_k S_k$ be a cuspidal divisor and set*

$$\rho_D = \exp\left(2\pi\left(\sum_j m_j^2 \cdot C_{jj} + \sum_{j\neq k} m_j m_k \left(C_{jj}+C_{kk}-C_{jk}\right)\right)\right).$$

If some non-zero multiple of D is principal, then there exist rational numbers $\gamma_\mathfrak{p}$ such that

$$\rho_D = \prod_{\mathfrak{p}\in\mathcal{P}_\Gamma\cup\mathcal{P}_S} N(\mathfrak{p})^{\gamma_\mathfrak{p}}.$$

Proof. If some non-zero multiple of D is a principal divisor then its Néron-Tate height vanishes. The claim then follows by exponentiating equation (6.12). □

Note, if the genus of $X(\Gamma)$ is zero, then every cuspidal divisor is principal. This gives non-trivial relations between the quantities C_{jk}.

7 Examples

7.1. Let us illustrate in an example how one could recover the Manin-Drinfeld theorem for the modular curves associated to congruence subgroups. Recall for any prime p the congruence subgroup $\Gamma_0(p)$ is defined by

$$\Gamma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \,\middle|\, c \equiv 0 \mod p \right\}.$$

The curve $X_0(p) := X(\Gamma_0(p))$ has two cusps denoted by S_∞ and S_0 of widths 1 and p respectively. Clearly $C(\Gamma_0(p))$ is torsion, since

$$\operatorname{div} \frac{\Delta(p\tau)}{\Delta(\tau)} = (p-1)(S_\infty - S_0). \tag{7.2}$$

Let us denote the cuspidal divisor $S_0 - S_\infty$ by D, then (7.2) implies

$$\langle D, D \rangle_{NT} = 0.$$

Let us see how our result reflects this fact. To ease notation we assume $p \equiv 1 \mod 12$. Our formula gives

$$\langle D, D \rangle_{NT} = \sum_{q \text{ prime}} \gamma_q \log q - 2\pi \left(2C_{\infty 0} - C_{\infty\infty} - C_{00} \right).$$

Recall, see e.g. [He], page 536, that the scattering matrix for $\Gamma_0(p)$ is given by the equality

$$\Phi_{\Gamma_0(p)}(s) = \frac{1}{p^{2s}-1} \begin{pmatrix} p-1 & p^s - p^{1-s} \\ p^s - p^{s-1} & p-1 \end{pmatrix} \cdot \frac{\widehat{\zeta_{\mathbb{Q}}}(2s-1)}{\widehat{\zeta_{\mathbb{Q}}}(2s)},$$

where $\widehat{\zeta_{\mathbb{Q}}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta_{\mathbb{Q}}(s)$ is the completed Riemann zeta function, satisfying $\widehat{\zeta_{\mathbb{Q}}}(s) = \widehat{\zeta_{\mathbb{Q}}}(1-s)$. With this description we determine the term that will be of interest to us and get

$$-2\pi \left(2C_{\infty 0} - C_{\infty\infty} - C_{00} \right) = -\frac{12}{p-1} \log p.$$

Let us now show that the rational numbers γ_q except γ_p vanish and that γ_p is indeed $12/(p-1)$.

The modular curve $X_0(p)$ has a semi-stable model $f : \mathcal{X}_0(p) \longrightarrow \operatorname{Spec}\mathbb{Z}$ defined over $\operatorname{Spec}\mathbb{Z}$. The only prime of bad reduction is the prime p. We now describe the semi-stable model of $X_0(p)$, see e.g. [DeRa]. The fiber above p contains two irreducible components F_∞ and F_0. Both cusps determine disjoint sections s_∞ and s_0 of f. The intersection matrix is given by

	F_0	F_∞	s_0	s_∞
F_0	$-\frac{p-1}{12}$	$\frac{p-1}{12}$	1	0
F_∞	$\frac{p-1}{12}$	$-\frac{p-1}{12}$	0	1

From this we get that the extension

$$\mathcal{D} = s_0 - s_\infty + \frac{12}{p-1} \cdot F_0.$$

of D is perpendicular to all the irreducible components of the fibers of f. By theorem 5.5 the Néron-Tate height of D is therefore given by

$$\begin{aligned}
-\langle D, D\rangle_{NT} &= \overline{\mathcal{O}(\mathcal{D})}.\overline{\mathcal{O}(\mathcal{D})} \\
&= \left(\frac{12}{p-1}\right)^2 \cdot \mathcal{O}(F_0).\mathcal{O}(F_0) + 2\frac{12}{p-1} \cdot \mathcal{O}(F_0).\left(\mathcal{O}(s_0) \otimes \mathcal{O}(s_\infty)^{-1}\right) \\
&\quad + \overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_0)} - 2\overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_\infty)} + \overline{\mathcal{O}(s_\infty)}.\overline{\mathcal{O}(s_\infty)} \\
&= \frac{12}{p-1}\log p \\
&\quad + \overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_0)} - 2\overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_\infty)} + \overline{\mathcal{O}(s_\infty)}.\overline{\mathcal{O}(s_\infty)}.
\end{aligned}$$

Applying proposition 6.1, we get since the cusp s_0 and s_∞ do never meet each other

$$\overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_\infty)} = -2\pi(C_{00} + C_{\infty\infty} - C_{0\infty}).$$

The natural morphism $\pi_\Gamma : X_0(p) \to X(1)$ extends to the semi-stable model and the cusps never meet $div(\pi_\Gamma^* j)^+$, therefore proposition 6.6 implies

$$\overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_0)} = -2\pi C_{00} \qquad \text{and} \qquad \overline{\mathcal{O}(s_0)}.\overline{\mathcal{O}(s_\infty)} = -2\pi C_{\infty\infty}.$$

Clearly summing all the terms gives zero.

7.3. Instructive examples of modular curves associated to non-congruence groups that occur in the literature are the Fermat curves $X^n + Y^n = 1$ (see [MR]) and the elliptic curves associated to subgroups of $\Gamma_0(16)$ considered by G. Berger [Ber], in both examples example $C(\Gamma)$ is known to be torsion. So let D be a cuspidal divisor, then corollary 6.13 gives that the associated linear combination of special values of Dirichlet series is the logarithm of some root of a rational number.

7.4. Remark. In general one may define non-congruence subgroups by generators and relations. This description does not give insights to the quantities we like to calculate. It seems to be a interesting problem to find an example, where one can calculate non-trivial Néron-Tate heights by the method proposed in this article.

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